The WAT Bézier Curves and Its Applications

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Abstract

In this paper, a kind of quasi-cubic Bézier curves by the blending of algebraic polynomials and trigonometric polynomials using weight method is presented, named WAT Bézier curves. Here weight coefficients are also shape parameters, which are called weight parameters. The interval [0, 1] of weight parameter values can be extended to $[-2,\pi^2/(\pi^2-6)]$. The WAT Bézier curves include cubic Bézier curves and C-Bézier curves ($\alpha=\pi$) as special cases. Unlike the existing techniques based on C-Bézier methods which can approximate the Bézier curves only from single side, the WAT Bézier curves can approximate the Bézier curve from the both sides, and the change range of shape of the curves is wider than that of C-Bézier curves. The geometric effect of the alteration of this weight parameter is discussed. Some transcendental curves can be represented by the introduced curves exactly.

Keywords

C-Bézier Curve; WAT Bézier Curve; Weight Parameter; Paths; Transcendental Curve

Introduction

It is well known that Bézier curves, in particular, the quadratic and cubic Bézier curves, are well known geometric modeling tools in CAGD. However, the shape of these curves can not be changed relative to their control polygon only if the user changes control polygon. For given control points, the weight numbers of the NURBS curves have an influence on the shape of the curves, however, adjusting the shape of a curve by changing the weight numbers is quite opaque to the user. On the other hand, since those curves are made up of algebraic polynomials, they have many shortcomings. For example, they can't represent transcendental curves such as the cycloid and helix etc. Hence, some new methods were proposed in the space of mixed algebraic and non-algebraic polynomials.

C-curves are extensions of the widely used cubic spline curves and are introduced by Zhang applying the basis $\{1,t,\sin t,\cos t\}$. In the case of C-B-splines, this extension coincides with the helix splines defined by Pottmann and Wagner. These tools provide exact representations of several important curves and surfaces such as the circle and the cylinder, the ellipse, the sphere, the cycloid and the helix. Further properties of C-curves have been studied by Mainar and Pena and Yang and Wang. C-curves are all defined on the interval $t\in(0,\alpha]$, where $\alpha\in(0,\pi]$ is a given real number. The change range of the C-curves is limited. Also, C-curves can only approximate the cubics from single side. On the other hand, since α appears in all the basis functions, it heavily affects the shape of the C-curves. Modifying one or more data of a given spline curve, the points of the curve will move on certain curves called paths. For example, moving one of the control points of a Bézier or B-spline curve, these paths will be parallel line segments, while changing a weight of a NURBS curve points of this curve will move towards the specified control point along line segments. Alteration of a knot value of a non-uniform B-spline curve yields well-defined rational curves as paths. If the parameter α of a C-curve is altered, the points of the curve obviously change their positions as well. However, as mentioned by Hoffmann, Paths of C-curves associated to $\alpha\neq\pi/2$ are not lines as one can easily observe by the mathematical extension of the paths.

In this paper, we present a class of quasi-cubic Bézier curves with weight parameter based on the blending space span $\{1,t,t^2,t^3,\sin\pi t,\cos\pi t\}$. The new curves include cubic Bézier curve and C-Bézier curve associated to $\alpha=\pi$ as special cases, and can approximate the cubic Bézier curve from the both sides. Also the change range of the curves is wider than that of C-Bézier curves. The paths of the given curves are line segments. Some transcendental curves can be represented by the WAT with the shape parameters and control points chosen properly.

The present paper is organized as follows. In Section 2, the basis functions based on the blending of Bernstein basis functions and trigonometric functions using weight method are established and the properties of the basis functions are shown. In Section 3, the WAT Bézier curves are given and some properties are discussed. Some transcendental curves can be represented precisely with the WAT Bézier curves in Section 4 and conclusion is made in Section 5.

WAT Bézier Base Functions

Consider the second form of the C-Bézier basis functions, see [5]:

$$Z_0(t,\alpha)=\frac{\alpha(1-t)-\sin\alpha(1-t)}{\alpha-\sin\alpha},$$
$$Z_1(t,\alpha)=M\left(\frac{1-\cos\alpha(1-t)}{1-\cos\alpha}-Z_0(t,\alpha)\right),$$
$$Z_2(t,\alpha)=M\left(\frac{1-\cos\alpha t}{1-\cos\alpha}-Z_3(t,\alpha)\right),$$
$$Z_3(t,\alpha)=\frac{\alpha t-\sin\alpha t}{\alpha-\sin\alpha},$$

Where

$$M=\begin{cases}1 & if\,\alpha=\pi\\ \dfrac{(1-\cos\alpha)\sin\alpha}{2\sin\alpha-\alpha(1+\cos\alpha)}, & otherwise\end{cases}$$

When $\alpha\to 0$, the basis functions are just cubic Bernstein basis,

$$\begin{cases}B_0(t)=(1-t)^3,\\ B_1(t)=3(1-t)^2t,\\ B_2(t)=3(1-t)t^2,\\ B_3(t)=t^3.\end{cases}\tag{2.1}$$

When $\alpha=\pi$, the basis functions are the following:

$$\begin{cases}T_0(t)=1-t-\dfrac{\sin\pi t}{\pi},\\ T_1(t)=-\dfrac{1}{2}+t+\dfrac{1}{2}\cos\pi t+\dfrac{\sin\pi t}{\pi},\\ T_2(t)=\dfrac{1}{2}-t-\dfrac{1}{2}\cos\pi t+\dfrac{\sin\pi t}{\pi},\\ T_3(t)=t-\dfrac{\sin\pi t}{\pi}.\end{cases}\tag{2.2}$$

Next, we will construct a new class of basis functions based on (2.1) and (2.2) using weight method.

Definition2.1 For $0\le\lambda\le 1$, the following four functions of $t\in[0,1]$ are defined as WAT Bézier basis functions,

$$\begin{cases}WAT_0(t,\lambda)=\lambda(1-t)^3+(1-\lambda)\dfrac{\pi(1-t)-\sin\pi t}{\pi},\\ WAT_1(t,\lambda)=3\lambda(1-t)^2t+(1-\lambda)\left(-\dfrac{1}{2}+t+\dfrac{1}{2}\cos\pi t+\dfrac{\sin\pi t}{\pi}\right),\\ WAT_2(t,\lambda)=3\lambda(1-t)t^2+(1-\lambda)\left(\dfrac{1}{2}-t-\dfrac{1}{2}\cos\pi t+\dfrac{\sin\pi t}{\pi}\right),\\ WAT_3(t,\lambda)=\lambda t^3+(1-\lambda)\dfrac{\pi t-\sin\pi t}{\pi}.\end{cases}\tag{2.3}$$

Obviously, WAT Bézier basis functions are cubic Bernstein bases when $\lambda=1$. And, when $\lambda=0$, WAT Bézier basis functions are C-Bézier bases associated to $\alpha=\pi$, see.

Straight calculation testifies that these WAT Bézier bases have the properties similar to the cubic Bernstein basis as follows.

1) *Properties at the endpoints:*

$$\begin{cases}WAT_0(0,\lambda)=1\\ WAT_3(1,\lambda)=1\end{cases},\quad\begin{cases}WAT_i^{(j)}(0,\lambda)=0\\ WAT_{i-3}^{(j)}(1,\lambda)=0\end{cases},\tag{2.4}$$

where $j=0,1,\cdots,i-1, i=1,2,3$, and $WAT_i^{(0)}(t,\lambda)=WAT_i(t,\lambda)$.

2) *Symmetry:*

$$\begin{cases}WAT_1(t,\lambda)=WAT_2(1\text{-}t,\lambda)\\ WAT_0(t,\lambda)=WAT_3(1\text{-}t,\lambda)\end{cases}\tag{2.5}$$

3) *Partition of unity:*

$$\sum_{i=0}^{3}WAT_i(t,\lambda)=1.\tag{2.6}$$

4) *Nonnegativity:*

$$WAT_i(t,\lambda)\ge 0,\ i=0,1,2,3.\tag{2.7}$$

According to the method of extending definition interval of C-curves in Ref., the interval [0, 1] of weight parameter values can be extended to $[-2,\frac{\pi^2}{\pi^2-6}]$, where $\frac{\pi^2}{\pi^2-6}\doteq 2.55055$.

WAT Bézier Curves

Construction of the WAT-Bézier Curves

Definition 3.1 Given points $P_i\,(i=0,1,2,3)$ in R^2 or R^3, then

$\boldsymbol{r}(t,\lambda)=\sum_{i=0}^{3}P_iWAT_i(t,\lambda)$, (3.1) is called a WAT Bézier curve, where $t\in[0,1]$, $\lambda\in[-2,2.55055]$, and $WAT_i(t,\lambda)(i=0,1,2,3)$ are the WAT Bézier basis.

From the definition of the base function, it is easy to find that WAT Bézier curves have the properties similar to the cubic Bézier curve and C-Bézier curve such as terminal properties, symmetry, geometric invariance and convex hull property.

Shape Control of the WAT-Bézier Curves

Due to the interval [0, 1] of weight parameter values can be extended to $[-2, 2.55055]$, the change range of the WAT-Bézier curve is wider than that of C-Bézier. From the Figure 1, it can be seen that when the control polygon is fixed, by adjusting the weight parameter from -2 to 2.55055, the WAT-Bézier curves can cross the cubic Bézier curves and reach the both sides of cubic Bézier curves, in other words, the WAT-Bézier curves can range from below the C-Bézier curve to above the cubic Bézier curve. The weight parameters have the property of geometry. The larger the shape parameter is, and the more approach the curves to the control polygon is. Also, these WAT-Bézier curves we defined include C-Bézier curve ($\alpha = \pi$) as special cases.

Paths of WAT-Bézier Curves

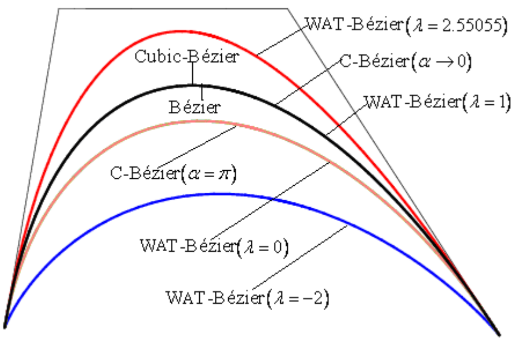

FIG. 1 ADJUSTING WAT-BÉZIER CURVES

As mentioned above, Modifying value of λ within the interval $\lambda \in [-2, \pi^2/(\pi^2-6)]$ yields a family of WAT-Bézier curves. For every fixed value of t, $\boldsymbol{r}(t,\lambda)$ represents a curve called path of WAT-Bézier curves with the running parameter. From Figure 2, it can be seen that path of WAT-Bézier curves is line segment, but path of C-Bézier curve is not.

In fact, when t is fixed, by computing the first derivative of $\boldsymbol{r}(t,\lambda)$ with respect to λ in (3.1), we get

$$d\boldsymbol{r}(t) = \sum_{i=0}^{3} \frac{d\left(WAZ_i(t,\lambda)\right)}{d\lambda} P_i d\lambda \qquad (3.2)$$

Where

$$\begin{cases} \dfrac{d\left(WAZ_0(t,\lambda)\right)}{d\lambda} = (1-t)^3 - \dfrac{\pi(1-t) - \sin\pi t}{\pi}, \\ \dfrac{d\left(WAZ_1(t,\lambda)\right)}{d\lambda} = 3(1-t)^2 t - \left(-\dfrac{1}{2} + t + \dfrac{1}{2}\cos\pi t + \dfrac{\sin\pi t}{\pi}\right), \\ \dfrac{d\left(WAZ_2(t,\lambda)\right)}{d\lambda} = 3(1-t)t^2 - \left(\dfrac{1}{2} - t - \dfrac{1}{2}\cos\pi t + \dfrac{\sin\pi t}{\pi}\right), \\ \dfrac{d\left(WAZ_3(t,\lambda)\right)}{d\lambda} = t^3 - \dfrac{\pi t - \sin\pi t}{\pi}. \end{cases}$$

Observing the right-hand side of equation (3.2, it can be found that $d\boldsymbol{r}(t,\lambda)$ is linear with respect to $d\lambda$ for fixed t, but for the C-Bézier curves, the $dQ(t,\alpha)$ is nonlinear with respect to $d\alpha$ for fixed $\boldsymbol{t}$. For example, let $Q(t,\alpha)$ be a C-Bézier curve and $z_{i,3}(t,\alpha)$ be base functions of C-Bézier curve. When $\boldsymbol{t}$ is fixed and α varies by $d\alpha$, set $d\alpha$, then

$$dQ(t,\alpha) = \sum_{i=0}^{3} \frac{dz_{i,3}(t,\alpha)}{d\alpha} \boldsymbol{q}_i d\alpha,$$

where

$$\frac{dz_{1,3}(t,\alpha)}{d\alpha} = \frac{(1-\cos\alpha)(3\sin\alpha - \alpha\cos\alpha - 2\alpha)}{2\left(\cos\alpha + \alpha\cos\dfrac{\alpha}{2} - 1\right)} \times$$
$$\left(\frac{1-\cos(\alpha-\alpha t)}{1-\cos\alpha} - z_{0,3}(t,\alpha)\right) + \frac{\sin\alpha(1-\cos\alpha)}{2\sin\alpha - \alpha - \alpha\cos\alpha} \times$$
$$\left(-\frac{(1-t)\sin(\alpha-\alpha t)(1-\cos\alpha) + (1-\cos(\alpha-\alpha t))\sin\alpha}{(1-\cos\alpha)^2} - \frac{dz_{0,3}(t,\alpha)}{d\alpha}\right)$$
$$(0 < \alpha < \pi)$$

$\dfrac{dz_{i,3}(t,\alpha)}{d\alpha} (\boldsymbol{i} = 0,2,3)$ is omitted.

So the WAT-Bézier curves have more advantages in shape adjusting than that C-Bézier curves do.

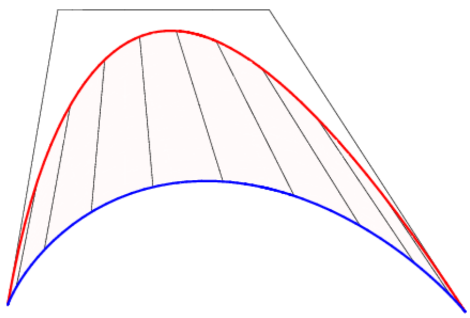

FIG. 2 PATHS OF WAT-BÉZIER CURVES

Representation of Some Transcendental Curves

In this section, some transcendental curves can represented with WAT-Bézier curves precisely.

Proposition 4.1 Let P_0, P_1, P_2 and P_3 be four control points. By proper selection of coordinates, their coordinates can be written in the form

$$P_0=(0,0), P_1=\left(\frac{1-\pi}{2}a,0\right), P_2=\left(\frac{1-\pi}{2}a,2a\right), P_3=(a,2a)\,(a\neq 0).$$

Then the corresponding WAT-Bézier curve with the weight parameters $\lambda=0$ and $t\in[0,1]$ represents an arc of cycloid.

Proof: If we take P_0, P_1, P_2 and P_3 into (3.1), then the coordinates of the WAT-Bézier curve are

$$\begin{cases} x(t)=a(t-\sin\pi t), \\ y(t)=a(1-\cos\pi t). \end{cases} \tag{4.1}$$

It is a cycloid in parametric form, see Figure3.

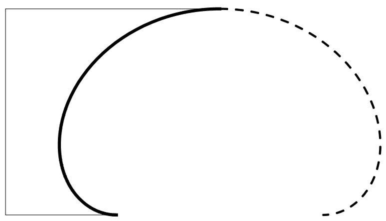

FIG. 3 THE REPRESENTATION OF CYCLOID WITH WAT-BÉZIER CURVE

Proposition 4.2 Let P_0, P_1, P_2 and P_3 be four properly chosen control points such that $P_0=(a,0,0), P_1=\left(0,a,\frac{\pi}{2}b\right), P_2=\left(-a,a,\frac{\pi}{2}b\right), P_3=(-a,0,b)\,(a\neq 0, b\neq 0)$.

Then the corresponding WAT-Bézier curve with the weight parameters $\lambda=0$ and $t\in[0,1]$ represents an arc of a helix.

Proof: Substituting P_0, P_1, P_2 and P_3 into (3.1) yields the coordinates of the WAT-Bézier curve

$$\begin{cases} x(t)=a\cos\pi t, \\ y(t)=a\sin\pi t, \\ z(t)=bt, \end{cases} \tag{4.2}$$

which is parameter equation of a helix, see Figure 4.

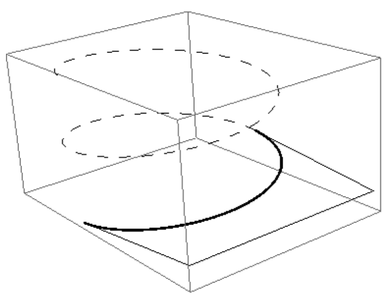

FIG.4.THE REPRESENTATION OF HELIX WITH WAT-BÉZIER CURVE

Proposition 4.3 Given the following four control points, $P_0=(0,0), P_1=P_2=\left(a,\frac{\pi}{2}b\right), P_3=(2a,0)\,(ab\neq 0)$.

Then the corresponding WAT-Bézier curve with the weight parameters $\lambda=0$ and $t\in[0,1]$ represents a segment of sine curve.

Proof: Substituting P_0, P_1, P_2 and P_3 into (3.1), we get the coordinates of the WAT-Bézier curve,

$$\begin{cases} x(t)=at, \\ y(t)=b\sin\pi t, \end{cases} \tag{4.3}$$

which implies that the corresponding WAT-Bézier curve represents a segment of sine curve, see Figure5.

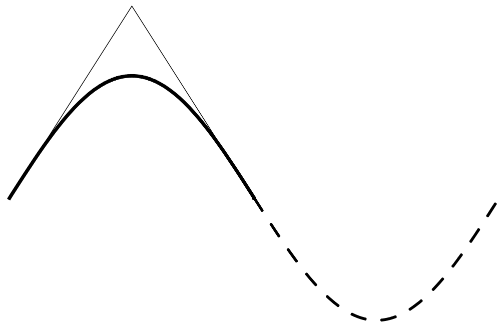

FIG.5 THE REPRESENTATION OF SINE CURVE WITH WAT-BÉZIER CURVES

Remark: If selecting proper control points and shape parameters, the cosine curve, the ellipse and the hyperbola can also be represented by WAT-Bézier curves.

Conclusions

In this paper, the WAT-Bézier curves based on the blending of algebraic polynomials and trigonometric polynomials have the similar properties that cubic Bézier curves have. The given curves can represent some special transcendental curves. What is more, the paths of the curves are linear, the WAT-Bézier curves have more advantages in shape adjusting than that C-Bézier curves.

Both rational methods (NURBS or Rational Bézier curves) and WAT-Bézier curves can deal with both free form curves and the most important analytical shapes for the engineering. However, WAT-Bézier curves are simpler in structure and more stable in calculation. The weight parameters of WAT-Bézier curves have geometric meaning and are easier to determine than the rational weights in rational methods. Meanwhile, WAT-Bézier curves can

represent the helix and the cycloid precisely, but NURBS can not. Therefore, WAT-Bézier curves would be useful for engineering.

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